UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part III Physics NST3PHY
MASTER OF ADVANCED STUDY IN PHYSICS MAPY

Thursday 27 April 2023 11.30 to 13.00

MINOR TOPICS
Paper 2/FSU (Formation of Structure in the Universe)

Answer **two** *questions only.*

The approximate number of marks allocated to each part of a question is indicated in the right-hand margin where appropriate.

The paper contains 10 sides including this one and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.

You should use a **separate Answer Book** *for each question.*

STATIONERY REQUIREMENTS
One yellow cover sheet
Two answer books
Rough workpad

SPECIAL REQUIREMENTS
Mathematical Formulae Handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

1 The James Webb Space Telescope (JWST) is searching for the first generation of stars (pop-III stars).

(a) From the dispersion relation, $a_T^2 k^2 - \omega^2 = 4\pi G\rho_0$, where $a_T = \sqrt{\frac{k_B T}{\mu}}$ is the isothermal sound speed with k_B the Boltzmann constant, T the temperature, μ the mass of the hydrogen atom, k the wavenumber, ω the angular frequency and G the gravitational constant, demonstrate that the critical mass above which gravity dominates for an isothermal sphere is given by :

$$M_J = \frac{\pi^{5/2}}{6} \frac{a_T^3}{G^{3/2}\rho_0^{1/2}}$$

where ρ_0 is the cloud density. [5]

(b) Recent simulations suggest that the temperature of H_2 clouds at z=30 is T=5000K with a density of ρ_0=6×10^{-12} kg/m^3 . Estimate the Jeans Mass for a star forming at z=30 in Solar Masses [1 $M_\odot$=2×10^{30}kg] [1]

(c) From Figure 1, estimate the luminosity of a pop-III star whose mass is $M_\star$ =300 $M_\odot$, its lifetime, radius and effective temperature. Models of formation of the first stars suggest that they have been formed in clumps whose total mass was ~10^{4}M$_\odot$. Estimate the luminosity of such a pop-III stars, assuming all stars have a stellar mass of $M_\star$ =300 $M_\odot$. [3]

(d) Assuming that pop-III stars are perfect thermal emitters, determine the wavelength at which their emission will be received on the *JWST* mirror. Using Figure 2, what can you conclude about the capabilities of the space telescope to observe such a clump of pop-III stars at z =30 ? [*The constant α in the Wien's displacement law is α=2.898×10^{-3} m.K*] [4]

(e) The formation of a pop-III star leads to the emission of a huge amount of UV photons which will ionise the neutral hydrogen formed at the recombination epoch. For simplicity, we can assume that the ionising photons create an ionised bubble surrounding the pop-III star. Demonstrate that the radius of this ionised sphere is given by :

$$R = \left(\frac{3}{4\pi}\frac{S_\star}{n_H^2 \alpha_B}\right)^{1/3}$$

where $S_\star$ is the total ionising photons radiated per second, α_B the recombination coefficient and n_H the density of neutral hydrogen. [4]

(f) Discuss briefly how these ionised bubbles evolve, and more generally how the fraction of neutral hydrogen evolves within the first billion years of the Universe. [3]

(a) *[seen - demonstrated during one of the lectures]* The limiting stability of the system is given by ω = 0, then when :

$$a_T^2 k_B^2 = 4\pi G\rho_0 \qquad (1)$$

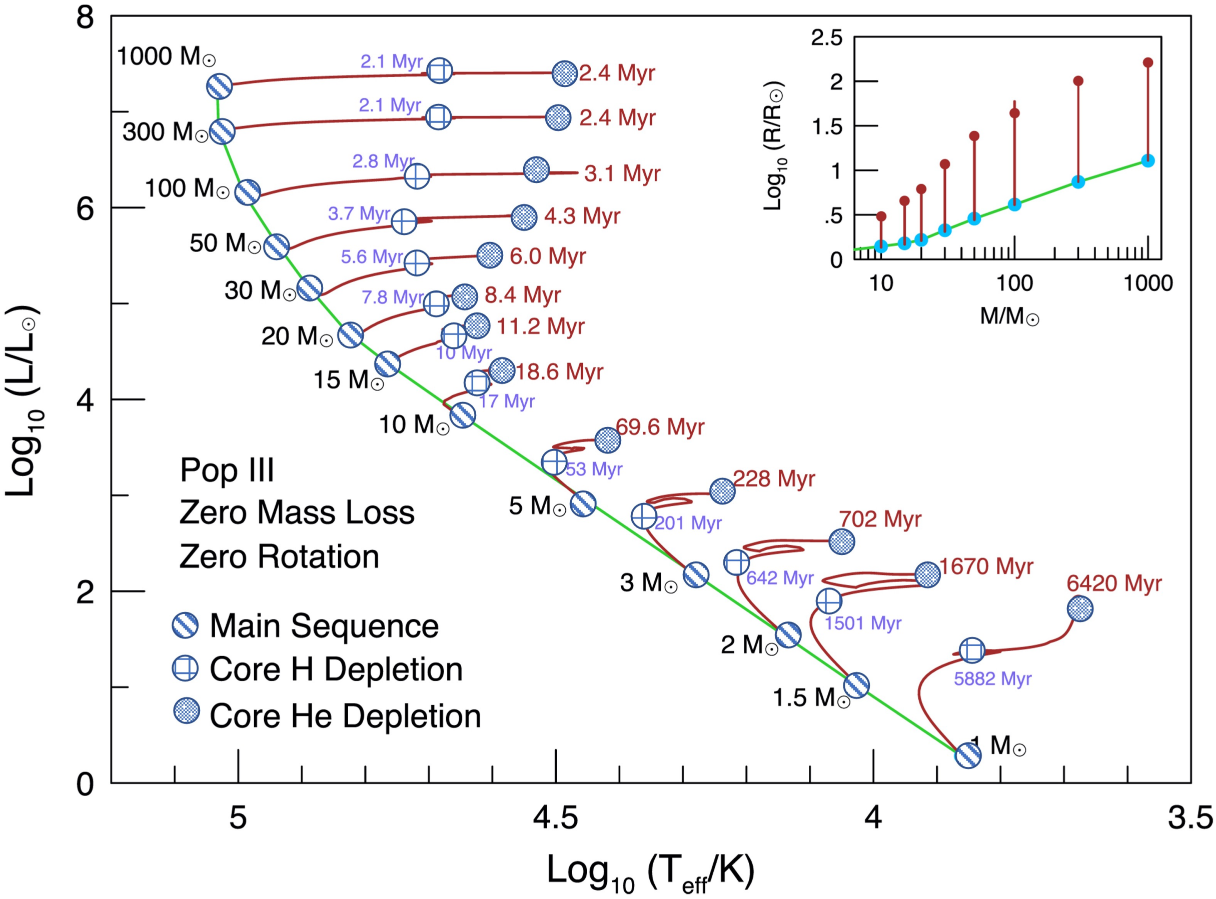

Figure 1: HR diagram for pop-III stars showing the evolution of their luminosities with different masses as a function of their effective temperatures. The lifetime of a star can be approximated by the time at which the He depletion start. From Windhorst et al. 2018, ApJS, 234, 41

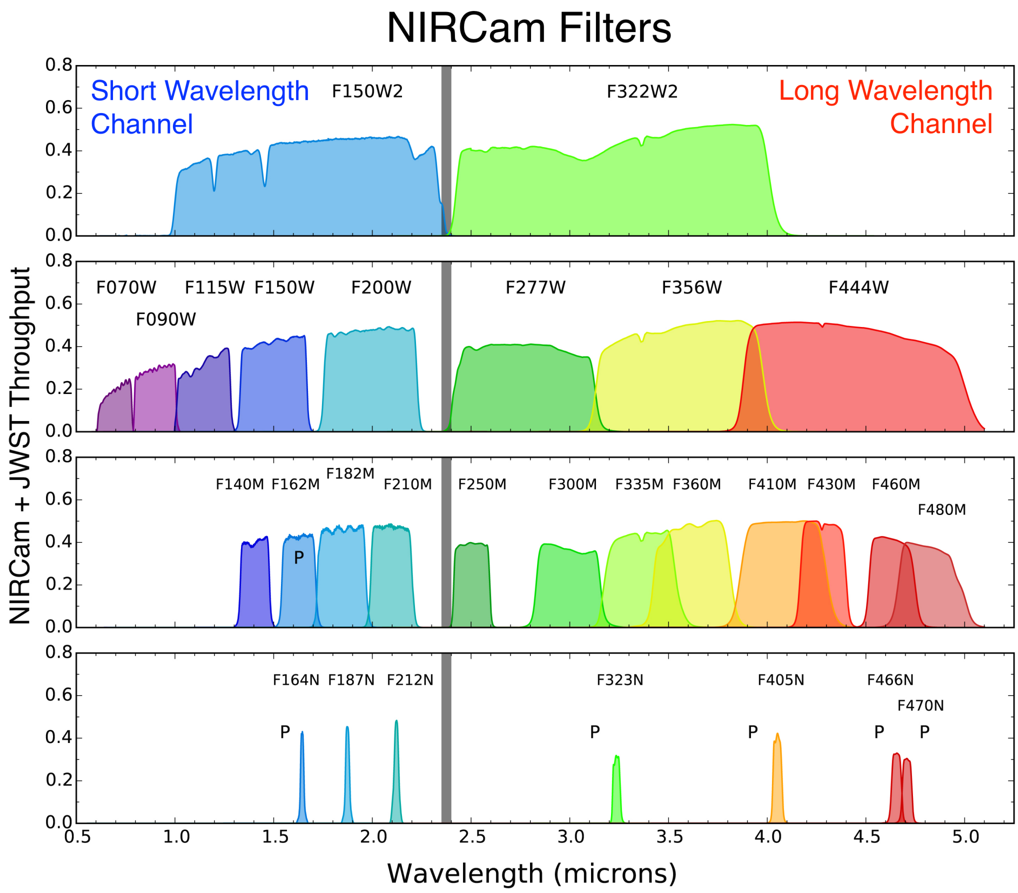

Figure 2: Total system throughput for each NIRCam filter. Throughput refers to photon-to-electron conversion efficiency. Averages of NIRCam modules A and B transmissions are plotted. The vertical gray bar marks the approximate dichroic cutoff between the short and long wavelength channels.

Hence, we can define the critical wavenumber as :

$$k_J^2 = \frac{4\pi G}{a_T^2}\rho_0 \tag{2}$$

At the limit of stability, the characteristic wavelength (also known as the Jeans wavelength) is :

$$\lambda_J = \frac{2\pi}{k_J} \tag{3}$$

The Jeans Mass is defined as the mass within a sphere of radius equal to the Jeans wavelength, such as :

$$M_J = \frac{4}{3}\pi\lambda_J^3\rho_0 \tag{4}$$

or :

$$M_J = \frac{4}{3}\pi\Big(\frac{2\pi}{k_J}\Big)^3\rho_0 = \frac{4}{3}\pi^4\Big(\frac{a_T}{2\pi^{1/2}\rho_0^{1/2}G^{1/2}}\Big)^3\rho_0 \tag{5}$$

which finally gives :

$$\boxed{M_J = \frac{\pi^{5/2}}{6}\frac{a_T^3}{\rho_0^{1/2}G^{3/2}}} \tag{6}$$

(b) Using the values of temperature and density given in the question, we obtain a Jeans Mass of 295 $M_\odot$

(c) *[unseen]* From Figure 1, we can estimate the luminosity of a pop-III star whose mass is 300$M_\odot$ at $L = 10^7 L_\odot$, a lifetime of 2.4Myr and a radius evolving from ~10 to 100 $R_\odot$. Assuming a total mass for a clump of $10^4 M_\odot$, therefore the number of star in that clumps is : 33, and the luminosity of the clump is 3.3×10^8 $L_\odot$.

(d) *[unseen]*According to the Wien's law :

$$\lambda_{peak} = \frac{\alpha}{T} \tag{7}$$

the peak wavelength of a pop-III stars with an effective temperature of ~10^5K (read on Figure 1) is 29nm, which is redshifted to 900nm, ie in the F090W NIRCam/JWST filter.

(e) *[bookwork]*The total ionising flux could be described as :

$$S_\star = n_p n_e \times \alpha_B \times V = n_p n_e \alpha_B \times \frac{4}{3}\pi R^3 \tag{8}$$

then the radius of the sphere, the so-called Strömgren sphere, assuming that n_H=n_p=$n_e$is given by :

$$R = \Big(\frac{3}{4\pi}\frac{S_\star}{n_H^2\alpha_B}\Big)^{1/3} \tag{9}$$

(f) *[discussed in one of the lectures]*The bubble's size increases, and eventually merges with other ionised bubbles. Planck observations show that the fraction of neutral hydrogen decreases strongly from z ~9 to z ~6. One billion years after the Big-Bang the hydrogen is mostly ionised.

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2 The James Webb Space Telescope (JWST) has observed bright systems of galaxies merging at $z \geq 6$.

(a) Demonstrate that the relaxation time scale for dynamical friction to act between two galaxies of mass M_1 and M_2 with a density of stars n is given by :

$$t_r = \frac{v^3}{4\pi G^2 M_1 M_2 n \ln \Lambda}$$

with G the gravitational constant, v the velocity, and $\Lambda = \frac{b_{max}}{b_{min}}$ the ratio of the extreme impact parameters. [6]

(b) Such a system has been observed by *JWST* in the GOODS-South Field at $z = 7.66$ with $\log(M_1\,[M_\odot]) = 8.96^{+0.12}_{-0.13}$ and $\log(M_2\,[M_\odot]) = 8.77^{+0.21}_{-0.21}$. Assuming $\ln \Lambda \sim 1$, $v \sim 200$ km/s and $n = 6\times10^{-7}\mathrm{pc}^{-3}$, estimate the relaxation time for this system. [*1pc = 3.086×10*^{16}*m*] [1]

(c) In a closed-box model developed in this course, the rate of metals production is given by :

$$\frac{d(gZ)}{dt} = (P + Z)(1 - \alpha)\Psi - Z\Psi$$

where Z is the metallicity, Ψ is the star formation rate of the galaxy, P is the yield (i.e. the production of new metals per unit of stellar mass), g the mass of gas and α the fraction of mass in stars. Demonstrate that the metallicity is given by :

$$Z = \frac{\wp}{\beta}\left(1 - \left(\frac{g}{M_0}\right)^{\frac{\beta}{1-\beta}}\right)$$

where $\wp$ is the yield, M_0 the total initial gas mass, and β a constant. You can assume that the infall rate is equal to $\alpha\beta\Psi$. [10]

(d) A protocluster of 6 galaxies at the exact same redshift (z=7.88) behind the lensing cluster Abell 2744 and distributed over a 60kpc radius region has been detected by the *JWST*. The total mass of this structure is $\sim 4\times10^{11}\mathrm{M}_\odot$. In a non-linear collapse model, the density of the protocluster is given by :

$$\rho_v = 200 \times \rho_0(1 + z_f)^3$$

where ρ_0 is the density at the formation epoch, and z_f the formation redshift. Estimate the formation redshift of this protocluster assuming $\rho_0 = 5\times10^8\mathrm{M}_\odot$ Mpc^{-3}. Discuss how the lensing cluster could affect the results. [3]

(a) *[similar to a bookwork question, but here the two masses are different]* We consider the interaction of two galaxies of mass M_1 and M_2, with velocity v_1 and v_2, each galaxy has change in perpendicular velocity given by :

$$\Delta v_\perp^1 = \frac{2GM_1}{bv} \tag{10}$$

$$\Delta v_\perp^2 = \frac{2GM_2}{bv} \tag{11}$$

Therefore the change in kinetic energy due to these changes in velocity is given by :

$$\Delta E_{k,\perp} = \frac{M_1}{2}\left(\frac{2GM_2}{bv}\right)^2 + \frac{M_2}{2}\left(\frac{2GM_1}{bv}\right)^2 = \frac{2G^2M_1M_2(M_1+M_2)}{b^2v^2} \tag{12}$$

This energy can only come from the forward motion of M_1, then :

$$\Delta E_{k,\perp} + \frac{M_1}{2}\Delta v_\parallel^2 = 0 \tag{13}$$

However :

$$\frac{1}{2}\Delta(v_\parallel^2) = v_\parallel \Delta v_\parallel \approx v\Delta v_\parallel \tag{14}$$

Then from eq. 13, we have :

$$-\Delta v_\parallel \approx \frac{\Delta E_{k,\perp}}{M_1 v_\parallel} = \frac{2G^2M_2(M_1+M_2)}{b^2v^3} \tag{15}$$

The number of collisions within $b \to b + db$ is : $vt \times 2\pi b db \times n$, and therefore integrating over all impact parameters :

$$-\frac{dv}{dt} = \int_{b_{min}}^{b_{max}} nv\frac{2G^2M_2(M_1+M_2)}{b^2v^3}2\pi b db = \frac{4\pi G^2(M_1+M_2)\rho}{v^2}\ln\Lambda \tag{16}$$

with $\rho = nM_2$ is the density within the interacting galaxy.
The relaxation time is obtain for $dv = v$ then :

$$\boxed{t_r = \frac{v}{dv/dt} = \frac{v^3}{4\pi G^2 M_1 \rho \ln\Lambda}} \tag{17}$$

Inserting numerical values gives t_r ~0.1 Myr

(b) *[demonstrated during one of the lectures]* In the case of no accretion, we have the following :

$$\frac{dg}{dt} = -\alpha\Psi \tag{18}$$

adding accretion to the previous equation gives :

$$\frac{dg}{dt} = -\alpha\Psi + \alpha\beta\Psi = \alpha(\beta-1)\Psi \tag{19}$$

The production of metals is given by :

$$\frac{d(gZ)}{dt} = (P+Z)(1+\alpha)\Psi - Z\Psi = P(1-\alpha)\Psi - \alpha Z\Psi \tag{20}$$

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then :

$$\frac{d(gZ)}{dt} = g\frac{dZ}{dt} + Z\frac{dg}{dt} = g\frac{dZ}{dt} + Z\alpha(\beta - 1)\Psi \quad (21)$$

Hence :

$$g\frac{dz}{dt} + Z\alpha(\beta - 1)\Psi = P(1 - \alpha)\Psi - \alpha Z\Psi \quad (22)$$

$$g\frac{dZ}{dt} = P(1 - \alpha)\Psi - \beta\alpha Z\Psi \quad (23)$$

or

$$\Psi = \frac{\frac{dg}{dt}}{\alpha(\beta - 1)} \quad (24)$$

Therefore :

$$g\frac{dZ}{dt} = \frac{P(1 - \alpha)}{\alpha}\frac{\frac{dg}{dt}}{(\beta - 1)} - \frac{\beta}{\beta - 1}Z\frac{dg}{dt} \quad (25)$$

defining $\wp$=$\frac{P(1-\alpha}{\alpha}$, then :

$$g\frac{dZ}{dt} = \frac{\wp - \beta Z}{\nu - 1}\frac{dg}{dt} \quad (26)$$

then :

$$\frac{dZ}{\wp - \beta Z} = \frac{1}{\beta - 1}\frac{dg}{g} \quad (27)$$

Integrating gives :

$$-\frac{1}{\beta}[\ln(\wp - \beta Z')]_0^Z = \frac{1}{\beta - 1}[\ln g]_{M_0}^g \quad (28)$$

Therefore :

$$\boxed{Z = \frac{\wp}{\beta}\left(1 - \left(\frac{g}{M_0}\right)^{\frac{\beta}{1-\beta}}\right)} \quad (29)$$

(c) *[bookwork]* We can easily estimate that :

$$(1 + z_f)^3 = \frac{\rho_v}{200\rho_0} \quad (30)$$

then

$$z_f = \left(\frac{\rho_v}{200\rho_0}\right)^{1/3} - 1 \quad (31)$$

Inserting the values gives : z_f =15.4. This protocluster has been detected behind a lensing cluster, therefore both luminosities (and hence masses) and volume of the clusters are affected. Therefore the total density of the cluster needs to be corrected for lensing magnification.

3 Dust grains in the interstellar medium (ISM) are mainly produced at the end of the life of massive stars.

(a) Assuming a Salpeter Initial Mass Function (α=-2.35), demonstrate that 16% of a galaxy stellar mass is in stars which will end their lives in type II SNe. [*You may assume that stellar masses are in the range $M_\star$=0.1 to 250$M_\odot$*]. [6]

(b) Dust grains in the ISM absorb part of the light emitted by stars. This absorption evolves with the distance R between the stars (luminosity L_ν)and the dust grain. Demonstrate that the temperature T of a dust grain with radius a is given by :

$$T = \left(\frac{L_\nu}{16\pi\sigma_B}\right)^{1/4}\left(\frac{1}{R^{1/2}}\right)$$

for the case of an optically thin dust cloud with perfect thermal absorption and emission by each grain, where σ_B is the Stefan-Boltzmann constant. [6]

(c) Dust grains sublimate at T=1500K. Consider dust grains distributed around an accreting black hole whose ultraviolet luminosity is 10^{11}L$_\odot$, assuming that each dust grain is perfect absorber and emitter. Estimate the minimum distance from the black hole at which the grains can survive. [2]

(d) Discuss briefly why a dust grain absorbs light in UV and re-emits in infra-red. Plot a galaxy SED from UV to sub-mm and indicate how the dust affect its shape. [6]

(a) *[unseen]* From the course, we know that stars with a mass above 8$M_\odot$ will end their lives as type II SNe.
The number of stars formed within the mass range m, m+dm is given by the integral of the Initial Mass Function ($\xi(m)$ hereafter). The total stellar mass of a galaxy is given by :

$$M_\star = \int_{M_{min}}^{M_{max}} m\xi(m)dm \tag{32}$$

Then the fraction of the galaxy stellar mass in stars with $M_\star \geq 8M_\odot$ is given by :

$$f_{M_\star}^{SNe} = \frac{\int_8^{250} m^{\alpha+1}dm}{\int_{0.1}^{250} m^{\alpha+1}dm} = \frac{[\frac{1}{\alpha+2}m^{\alpha+2}]_8^{250}}{[\frac{1}{\alpha+2}m^{\alpha+2}]_{0.1}^{250}} \tag{33}$$

Assuming a Salpeter IMF with α=-2.35, we obtain :

$$f_{M_\star}^{SNe} = \frac{[-\frac{1}{0.35}m^{-0.35}]_8^{250}}{[-\frac{1}{0.35}m^{-0.35}]_{0.1}^{250}} = \frac{250^{-0.35} - 8^{-0.35}}{250^{-0.35} - 0.1^{-0.35}} = 16.1\% \tag{34}$$

(b) *[bookwork]* The total power absorbed by a grain is given by :

$$P_{abs} = \frac{L_\nu}{4\pi R^2}\pi a^2 \tag{35}$$

In the case of a thermal emitter, the dust grain emits like a black body and the power emitted is given by :

$$P_{em} = 4\pi a^2\sigma_B T^4 \tag{36}$$

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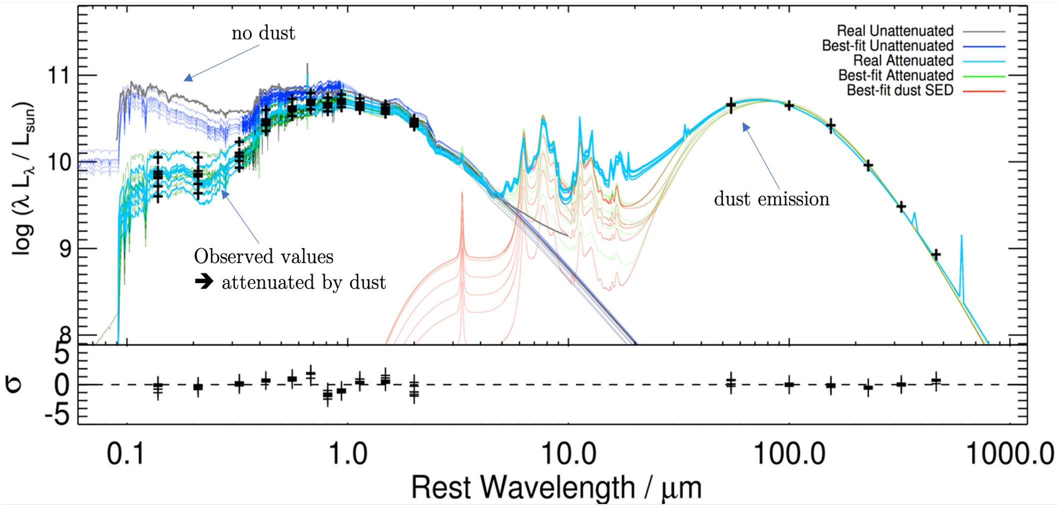

Figure 3: Influence of dust on the SED of a galaxy

At the equilibrium, we therefore have :

$$P_{abs} = P_{em} \tag{37}$$

then :

$$\frac{L_\nu}{4\pi R^2}\pi a^2 = 4\pi a^2 \sigma_B T^4 \tag{38}$$

and finally :

$$\boxed{T = \left(\frac{L_\nu}{16\pi\sigma_B}\right)^{1/4}\left(\frac{1}{R^{1/2}}\right)} \tag{39}$$

(c) *[unseen]* From the previous equation, we can easily get the radius :

$$\boxed{R = \frac{1}{T^2}\sqrt{\frac{L_\nu}{16\pi\sigma_B}}} \tag{40}$$

Replacing T=1500K, we obtain a value of 0.05pc

(d) *[bookwork]* • The size of dust grains is ranging from 1nm to 1μm, with a mean size of 0.1μm.
•As a consequence, the dust mostly absorb UV light. When a dust grain absorbs a UV photon, the energy from the photon is transferred to the molecules within the grain, causing them to vibrate and become excited and hence increasing its temperature.
•The dust grains in space are typically at temperatures of tens to hundreds of Kelvin, which means that the majority of their emitted radiation falls in the IR part of the spectrum (see the Wien's law displacement law).

END OF PAPER