
Gravitational collapse
Chapter 4

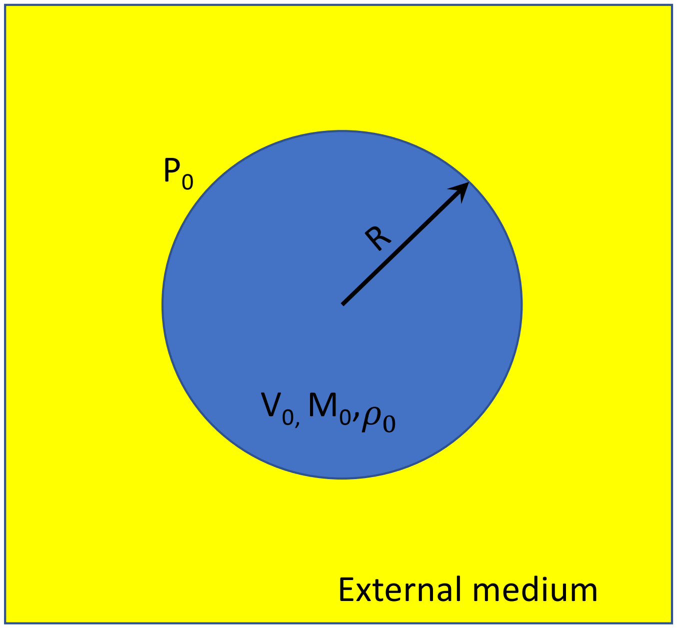
Free-fall time

The free-fall time is the characteristic time that would take an object to collapse under its own gravity, if no other forces exist to oppose the collapse.

We will first consider the collapse of a cloud within a galaxy, but this method applies also at larger scales (i.e., galactic scale)

INITIAL CONDITIONS OF THE COLLAPSE

- Cloud of radius R and mass M_0
- Gas density ρ_0
- Gas molecules initially at r_0 will have a mass of gas M within the radius and during collapse this remains constant



From Newton gravity the equation of motion is :

$$\frac{\partial^2 r}{\partial t^2} = -\frac{GM_r}{r^2}$$

or

$$\frac{\partial}{\partial t}\left(\frac{\partial r}{\partial t}\right) = -\frac{GM_r}{r^2}$$

Free-fall time

Multiplying by $\frac{\partial r}{\partial t}$ and integrating over dt give :

$$\frac{1}{2}\left(\frac{\partial r}{\partial t}\right)^2 = \left[\frac{GM}{r}\right]_{r_0}^{r} = \frac{GM}{r_0}\left(\frac{r_0}{r} - 1\right) = \frac{4\pi}{3} r_0^2 \rho_0 G \left(\frac{r_0}{r} - 1\right)$$

Then :

$$\frac{\partial r}{\partial t} = \sqrt{\frac{8\pi}{3} r_0^2 G \rho_0 \left(\frac{r_0}{r} - 1\right)}$$

Hence :

$$\partial t = \sqrt{\frac{3}{8\pi r_o^2 G \rho_0}} \frac{\partial r}{\sqrt{\frac{r_0}{r} - 1}}$$

Integrating :

$$t = \left(\frac{3}{8\pi r_0^2 G \rho_0}\right)^{1/2} \int_0^{r_0} \frac{\partial r}{\sqrt{\frac{r_0}{r} - 1}}$$

Substituting $u = r/r_0$ gives :

$$t = \left(\frac{3}{8\pi G \rho_0}\right)^{1/2} \int_0^{1} \frac{\partial u}{\sqrt{\frac{1}{u} - 1}}$$

Tabulated ➔ $\frac{\pi}{2}$

Therefore, the free-fall time is given by :

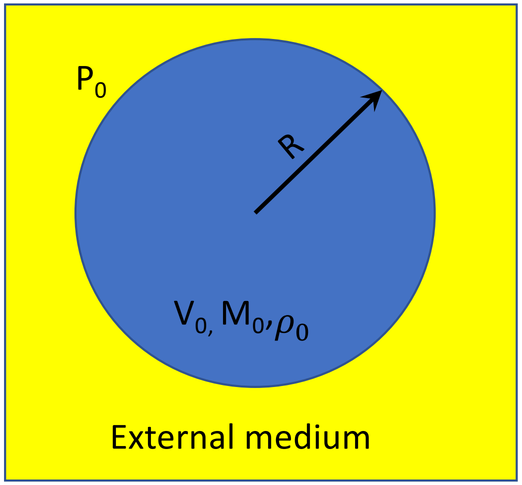
$$t_{ff} = \left(\frac{3\pi}{32 G \rho_0}\right)^{1/2}$$

The free-fall time depends on density : if the inner region of the cloud are denser, these will collapse first➔ **inside-out collapse**

Inside-out collapse

Adding to the previous initial conditions that the cloud has an isothermal behavior.

We also assume that there is a central sink for inflowing material (the growing central object: e.g., a protostar)



The dynamic of the problem is governed by the Euler's equation and the continuity equation (radial equations) :

$$\rho\frac{d\vec{v}}{dt} + \rho(\vec{v}\cdot\nabla)\vec{v} = -\nabla P - \rho\nabla\Phi_g$$

$$\frac{dv}{dt} + v\frac{dv}{dr} = -\frac{a_T^2}{\rho}\frac{d\rho}{dr} - \frac{GM_r}{r^2}$$

and

$$\frac{d\rho}{dt} + \frac{1}{r^2}\frac{d(r^2\rho v)}{dr} = 0$$

$$\frac{d\rho}{dt} + \nabla.(\rho v) = 0$$

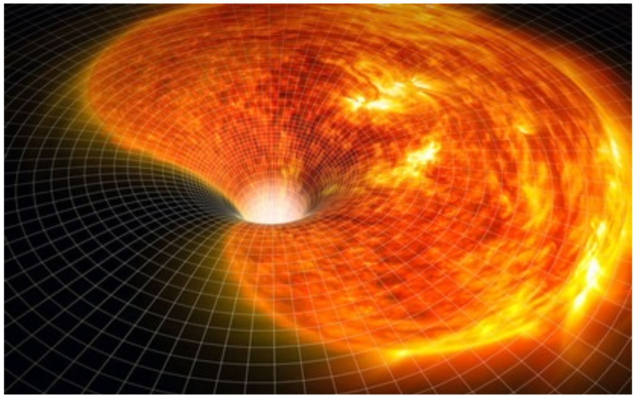
Divergence in spherical coordinates

$$\nabla_r \cdot f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)$$

where $\rho = \rho(r,t)$ and

$$M_r(t) = \int_0^r 4\pi r^2 \rho(r,t)dr$$

Differentiating gives : $\frac{\partial M_r}{\partial t} = -4\pi r^2 \rho v$



Inside-out collapse

$$\frac{dv}{dt} + v\frac{dv}{dr} = -\frac{a_T^2}{\rho}\frac{d\rho}{dr} - \frac{GM_r}{r^2}$$

$$\frac{d\rho}{dt} + \frac{1}{r^2}\frac{d(r^2\rho v)}{dr} = 0$$

SIMILARITY ANALYSIS

In the previous equations we can identify :

- The independent variables : r and t
- The constants : G and a_T
- The variables : $\rho(r,t), v(r,t)$ and $M(r,t)$

The only way to form a dimensionless length is :

$$x = \frac{r}{a_T t}$$

We are doing a similarity analysis; therefore, we are searching for solutions in the form :

$$M_r(r,t) = \frac{a_T^3 t}{G} m(x)$$

$$\rho(r,t) = \frac{1}{4\pi G t^2}\alpha(x)$$

$$v(r,t) = a_T \beta(x)$$

$m(x), a(x)$ and $\beta(x)$ are dimensionless

Then :

$$dx = \frac{1}{a_T t}dr - \frac{r}{a_T\, t^2}dt$$

$$\left(\frac{\partial}{\partial r}\right)_t = \frac{1}{a_T t}\frac{\partial}{\partial x}$$

$$\left(\frac{\partial}{\partial t}\right)_r = \left(\frac{\partial}{\partial t}\right)_x + \left(\frac{\partial x}{dt}\right)_r \left(\frac{\partial}{\partial x}\right) = \left(\frac{\partial}{\partial t}\right)_x - \frac{x}{t}\left(\frac{\partial}{\partial x}\right)$$

Inside-out collapse

SIMILARITY ANALYSIS

The equations we must solve are :

$$\frac{dv}{dt} + v\frac{dv}{dr} = -\frac{a_T^2}{\rho}\frac{d\rho}{dr} - \frac{GM_r}{r^2}$$

$$\frac{d\rho}{dt} + \frac{1}{r^2}\frac{dr^2\rho v}{dr} = 0$$

$$\frac{\partial M}{\partial r} = 4\pi r^2 \rho$$

Knowing that :

$$\left(\frac{\partial}{\partial r}\right)_t = \frac{1}{a_T t}\frac{\partial}{\partial x}$$

$$\left(\frac{\partial}{\partial t}\right)_r = \left(\frac{\partial}{\partial t}\right)_x + \left(\frac{\partial x}{dt}\right)_r \left(\frac{\partial}{\partial x}\right) = \left(\frac{\partial}{\partial t}\right)_x - \frac{x}{t}\left(\frac{\partial}{\partial x}\right)$$

The equations become :

$$m = x^2\alpha(x-\beta)$$

$$[(x-\beta)^2 - 1]\frac{1}{\alpha}\frac{d\alpha}{dx} = \left[\alpha - \frac{2}{x}(x-\beta)\right](x-\beta)$$

$$[(x-\beta)^2 - 1]\frac{d\beta}{dx} = \left[\alpha(x-\beta) - \frac{2}{x}\right](x-\beta)$$

These equations must be solved numerically, but we can learn a lot from their form.

Inside-out collapse

SIMILARITY ANALYSIS

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$$[(x-\beta)^2-1]\frac{d\beta}{dx} = \left[\alpha(x-\beta) - \frac{2}{x}\right](x-\beta)$$

We are looking for solutions with the form :

$$M_r(r,t) = \frac{a_T^3 t}{G} m(x)$$

$$\rho(r,t) = \frac{1}{4\pi G t^2}\alpha(x)$$

$$v(r,t) = a_T\beta(x)$$

An exact solution is the singular isothermal sphere, where we demonstrated that :

$$\mathrm{M(r_0)} = \frac{2a_T^2 r_0}{G} = \frac{a_T^3 t}{G}\,2x$$

then $m = 2x$, hence :

$$m = 2x = x^2\alpha(x-\beta)$$

or

$$\alpha = \frac{2}{x(x-\beta)}$$

and $\beta = \frac{v(r,t)}{a_T}$. In the singular isothermal sphere, the system is in equilibrium ($v(r,t) = 0$) then $\beta = 0$ and

$$\alpha = \frac{2}{x^2}$$

Inside-out collapse

SIMILARITY ANALYSIS

$$m = x^2\alpha(x-\beta)$$

$$[(x-\beta)^2-1]\frac{1}{\alpha}\frac{d\alpha}{dx} = \left[\alpha - \frac{2}{x}(x-\beta)\right](x-\beta)$$

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$$v(r,t) = a_T\beta(x)$$

Another singular solution is given by :

$$x-\beta = 1$$
$$\alpha = \frac{2}{x}$$

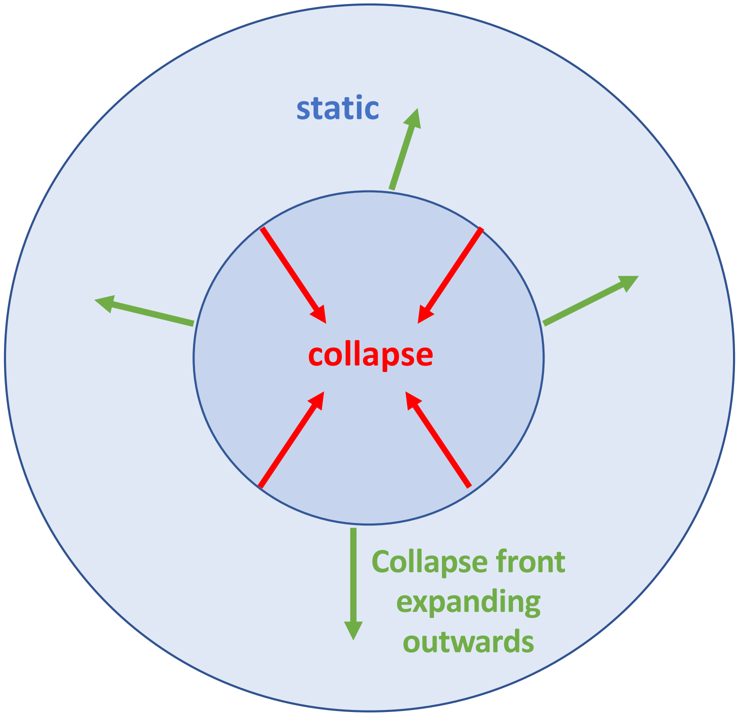
$x = 1$ is the crucial transition point :

- At $x > 1$: the solution is the singular isothermal sphere
- At $x < 1$: then $\beta < 0$, hence $v(r,t) < 0$ ➔ **infall**

The transition critical point between infall and static isothermal solution $(x_c = 1)$ translates into $r_c = a_T t$

This is a wave moving outwards at the sound speed a_T

Inside-out collapse



Inside-out collapse

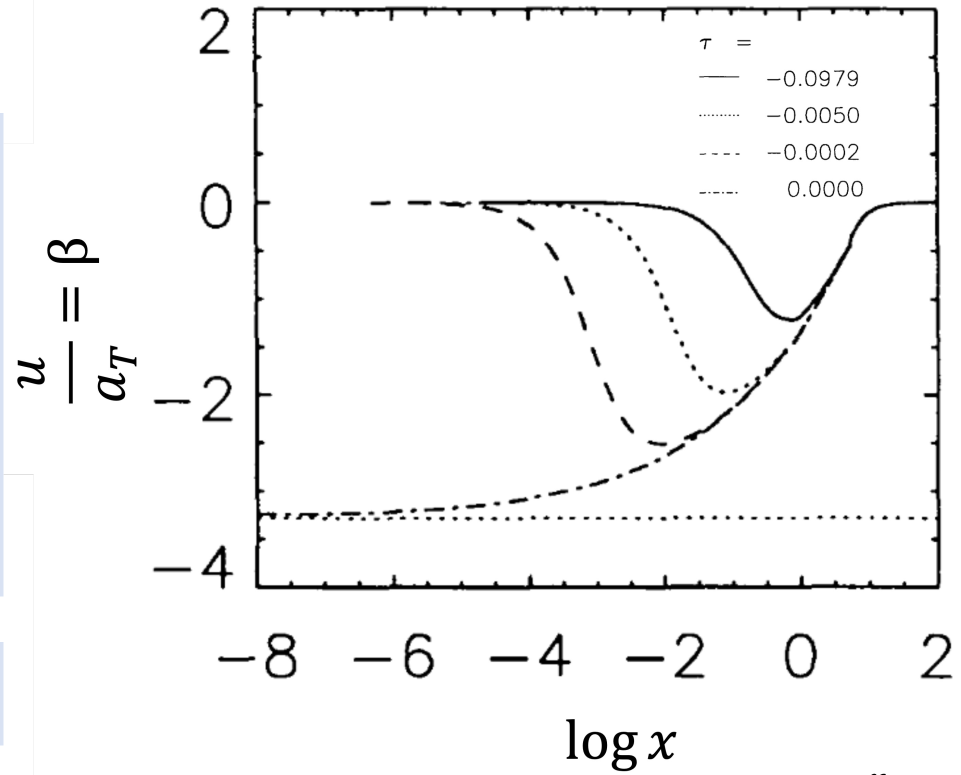
To do the previous analysis, we assumed that :

- The system is the equilibrium isothermal sphere
- The boundary conditions are those for that initial state
- There is a sink for matter reaching the origin : this will turn into a protostar

Another interesting case to consider is that of a cloud which is marginally unstable, for example with a mass slightly larger than the Bonnor-Ebert mass. We then perturbate the system and follow the evolution.

τ=0 : start of the creation of the protostar as mass starts to flow into the sink

<u>Example of numerical solution </u>: *velocity during the collapse of an isothermal sphere with mass slightly above the Bonnor-Ebert mass*



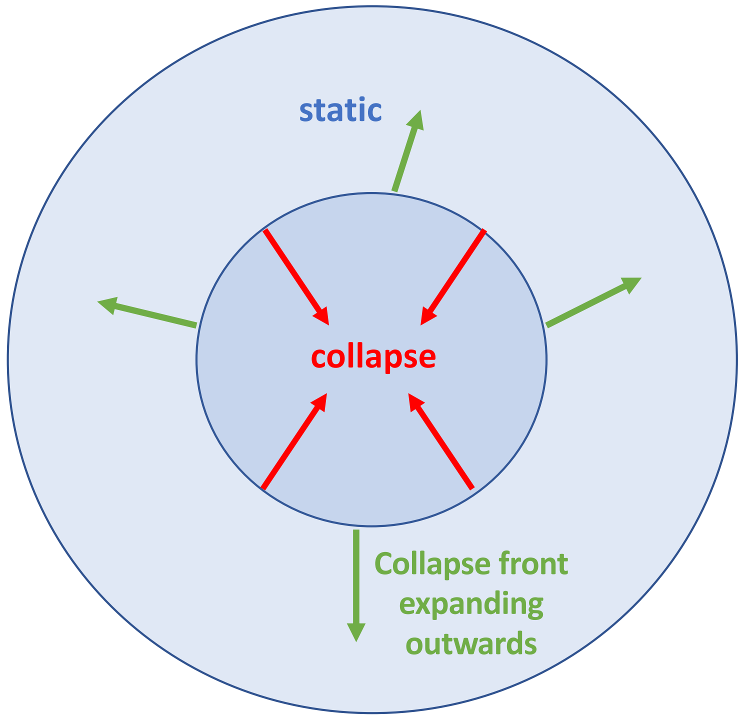
$$x = \frac{r}{a_T t}$$

Physics Analysis

The transition point moves outwards as a rarefaction wave with only the gas inside of the radius $R_{ff} \approx a_T t$ moving inward.

After a short fraction of a free-fall time a large fraction of the gas within this radius is moving supersonically with the velocity increasing to the centre :

$r/|v|$ is less than the sound crossing time.



Gas is falling onto a growing central object, the protostar, with a mass M_* :

- close to this protostar, gas is approximately in free fall

- $v_{ff} \approx \left(\frac{2GM_*}{r}\right)^{1/2}$

$$\frac{1}{2}v_{ff}^2 = \frac{GM}{r}$$

At the transition point, the gas moves approximately sonically :

- $v_{ff} \approx a_T$

$$r = a_T t$$

- $a_T^2 \sim M_* G / R_{ff}$

Physics Analysis

The rate of growth of M_* is determined by accretion at a rate :

$$\frac{dM}{dt} = \lim_{r \to 0} -4\pi r^2 v \rho$$

Assuming constant accretion rate $M_* = \frac{dM}{dt} t$ and

$$\frac{dM_*}{dt} \approx \frac{M_*}{t} \approx \frac{a_T^2}{G}\frac{R_{ff}}{t} \approx \frac{a_T^3}{G}$$

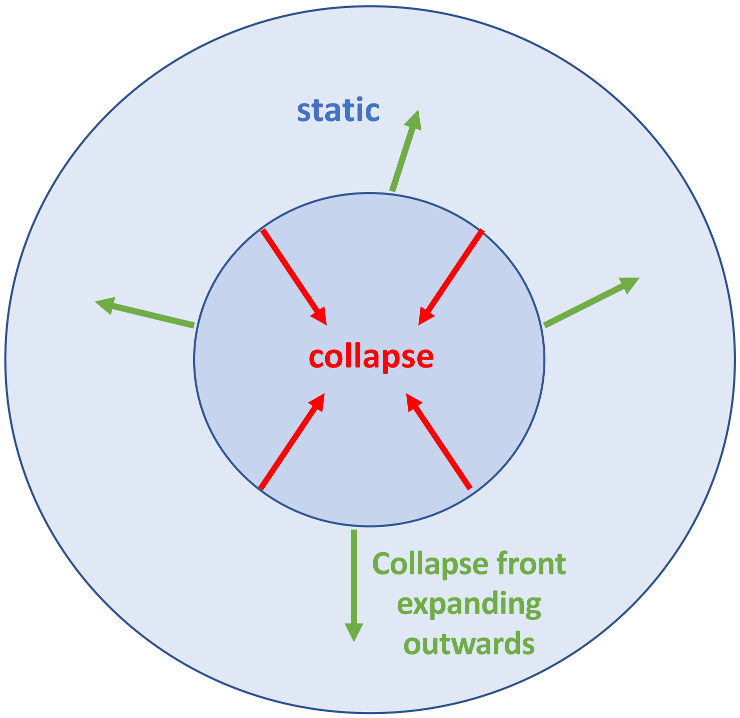
$a_T^2 \sim M_* G / R_{ff}$

$R_{ff} \approx a_T t$

Inserting values, the accretion rate for the growth of the protostar is :

$$\frac{dM_*}{dt} \approx 2\times 10^{-6} \left(\frac{T}{10K}\right)^{\frac{3}{2}} M_\odot \, yr^{-1}$$

$$a_T^2 = \frac{k_B T}{\mu}$$

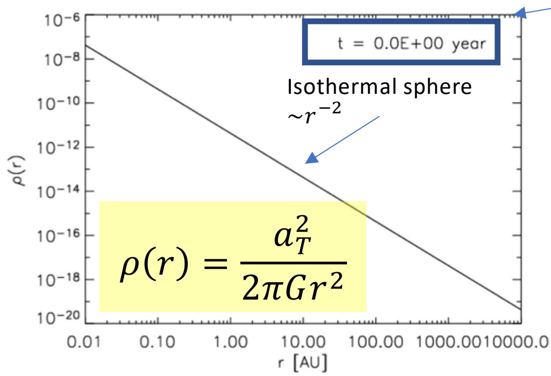


The density profile in the collapse region must satisfy :

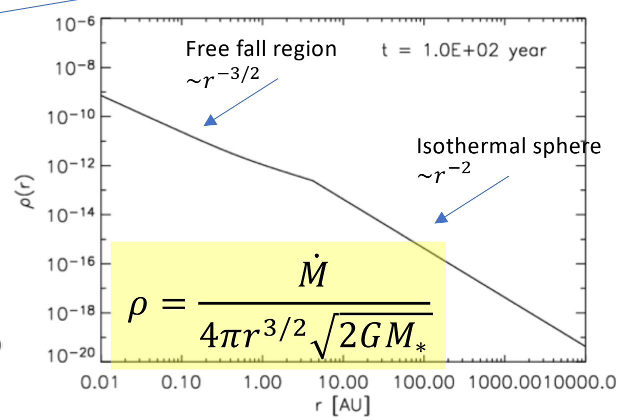
$$\rho = \frac{\dot{M}}{4\pi r^2 |v|} = \frac{\dot{M}}{4\pi r^2 v_{ff}} = \frac{\dot{M}}{4\pi r^{3/2}\sqrt{2GM_*}}$$

Physics Analysis

The collapse starts at t=0



$$\rho(r) = \frac{a_T^2}{2\pi G r^2}$$

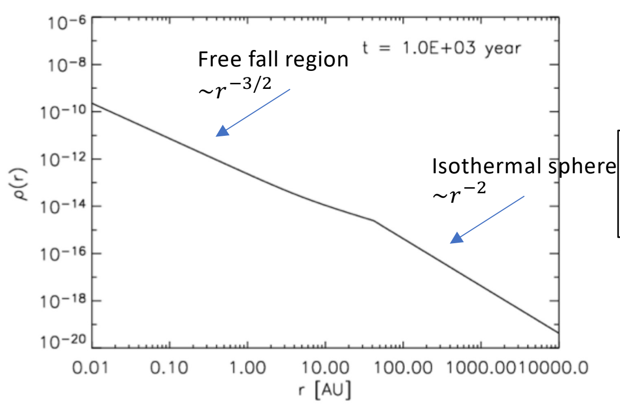


$$\rho = \frac{\dot{M}}{4\pi r^{3/2}\sqrt{2GM_*}}$$

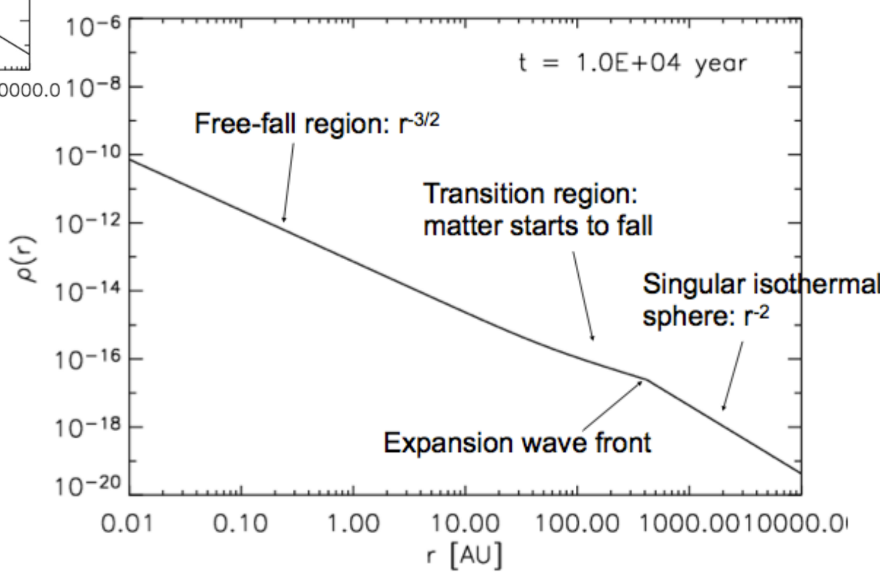
Since the Jean Mass is $M_j \propto \rho^{-1/2}$ inner regions have smaller Jeans Mass

➔ smaller regions will collapse into smaller sub-clumps

➔ **fragmentation**



Evolution of the density profile with time



Summary of the formation of structures in the Universe

FROM THE FIRST LECTURE

$$\frac{M_J}{M_\odot} = 1.0\times\left(\frac{T}{10K}\right)^{3/2}\times\left(\frac{n_H}{2\times10^{10}m^{-3}}\right)^{-1/2}$$

Gas clouds with multi-phases

dense regions

Low-density regions

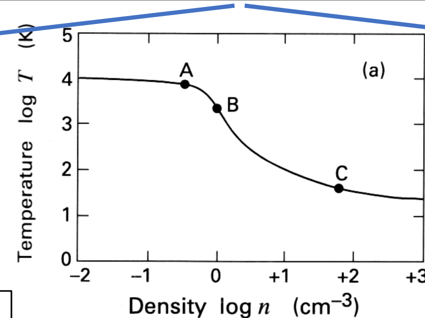
fast cooling → collapse

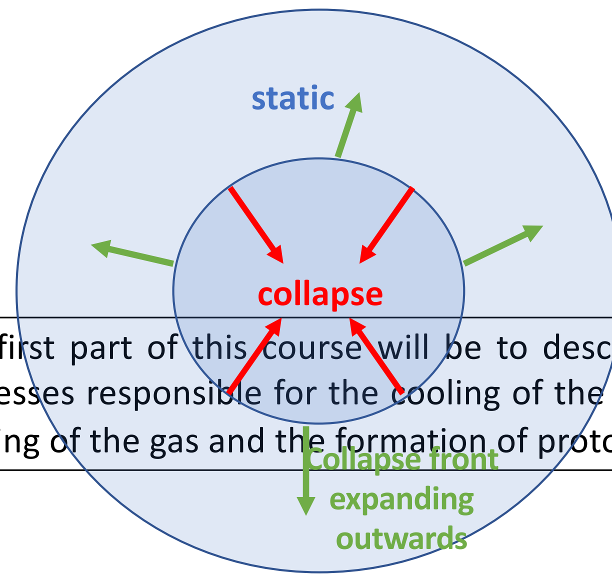
Gas clouds | Gas clouds | Gas clouds

$$t_{ff} = \left(\frac{3\pi}{32G\rho_0}\right)^{1/2}$$

Heating of collapsing gas

Protostar | Protostar | Protostar





The first part of this course will be to describe the processes responsible for the cooling of the gas, the heating of the gas and the formation of protostars.

Summary of Monday's lecture

We defined the free-fall time as the characteristic time that would take an object to collapse under its own gravity

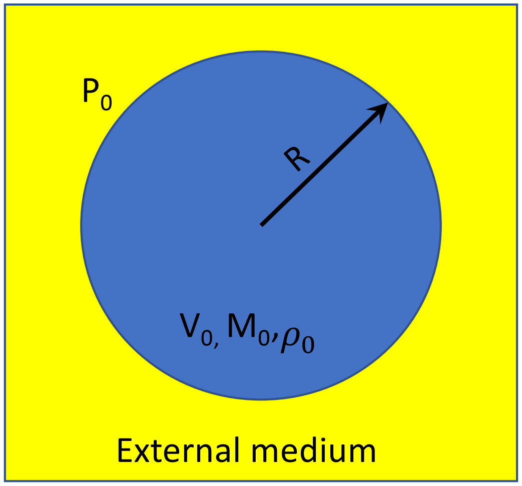
$$t_{ff} = \left(\frac{3\pi}{32G\rho_0}\right)^{1/2}$$

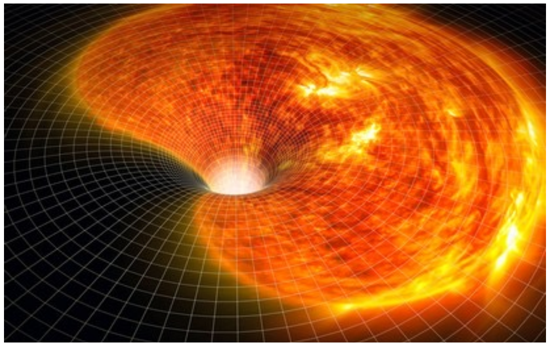
Only depends on the density, suggesting that denser region will collapse first

➔ inside-out collapse

$$\rho(r) = \frac{a_T^2}{2\pi G r^2}$$

Summary of Monday's lecture





We studied the case of a collapsing gas cloud with a sink at the center for the inflowing material, and do a similarity analysis to solve the Euler's and continuity equations

$$\frac{dv}{dt} + v\frac{dv}{dr} = -\frac{a_T^2}{\rho}\frac{d\rho}{dr} - \frac{GM_r}{r^2}$$

$$\frac{d\rho}{dt} + \frac{1}{r^2}\frac{d(r^2\rho v)}{dr} = 0$$

Dimensionless variable

$$x = \frac{r}{a_T t}$$

We were looking for solutions with the form :

$$M_r(r,t) = \frac{a_T^3 t}{G} m(x)$$

$$\rho(r,t) = \frac{1}{4\pi G t^2}\alpha(x)$$

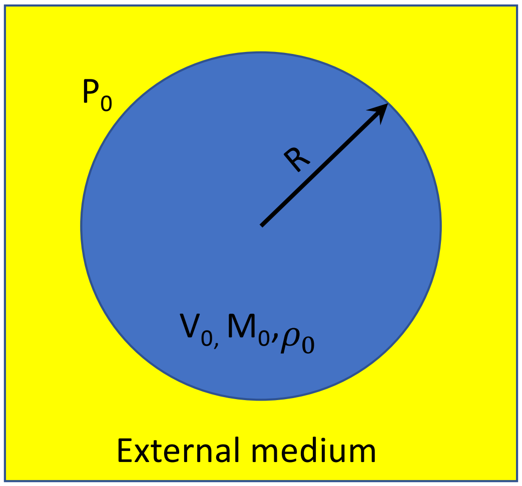
$$v(r,t) = a_T \beta(x)$$

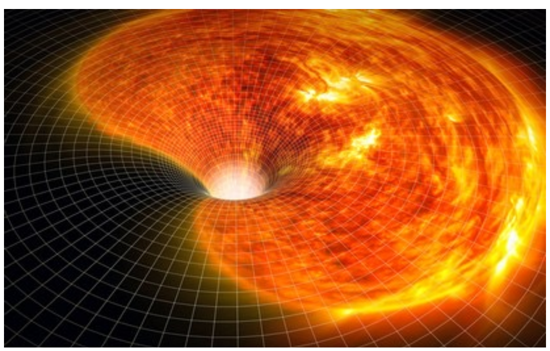
$$m = x^2\alpha(x-\beta)$$

$$[(x-\beta)^2 - 1]\frac{1}{\alpha}\frac{d\alpha}{dx} = \left[\alpha - \frac{2}{x}(x-\beta)\right](x-\beta)$$

$$[(x-\beta)^2 - 1]\frac{d\beta}{dx} = \left[\alpha(x-\beta) - \frac{2}{x}\right](x-\beta)$$

Summary of Monday's lecture





One exact solution of previous solution is the isothermal sphere, for which we found :

$$\mathrm{M}(\mathrm{r}_0) = \frac{2a_T^2 r_0}{G} = \frac{a_T^3 t}{G} 2x$$

$$M_r(r,t) = \frac{a_T^3 t}{G} m(x)$$

$$x = \frac{r}{a_T t}$$

$$m(x) = 2x$$

$$m(x) = x^2 \alpha(x - \beta)$$

$$\alpha = \frac{2}{x(x-\beta)}$$

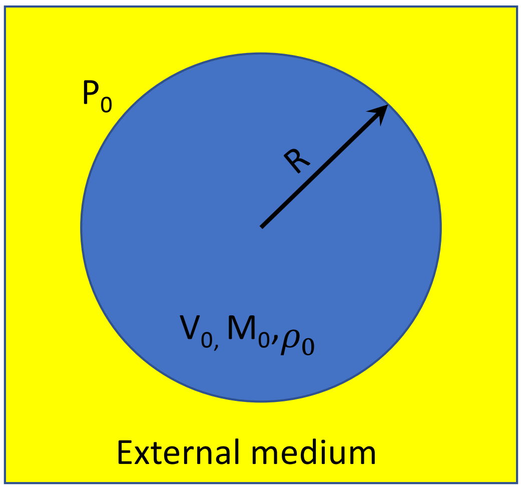
We also defined : $v(r,t) = a_T \beta(x)$

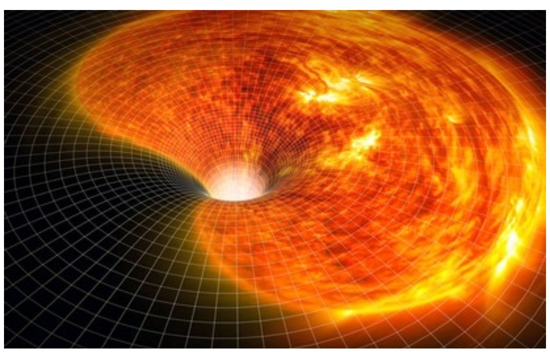
In the case of the singular isothermal sphere, the system is in equilibrium : $v(r,t) = 0 \rightarrow \beta = 0$

$$\alpha = \frac{2}{x^2}$$

$$\rho(r,t) = \frac{1}{2\pi G t^2 x^2} = \frac{a_T^2}{2\pi G r^2}$$

Summary of Monday's lecture





Another singular solution is obtained when $x - \beta = 1$

$$m = x^2 \alpha (x - \beta)$$

$$[(x-\beta)^2 - 1] \frac{1}{\alpha} \frac{d\alpha}{dx} = \left[\alpha - \frac{2}{x}(x-\beta)\right](x-\beta)$$

$$[(x-\beta)^2 - 1] \frac{d\beta}{dx} = \left[\alpha(x-\beta) - \frac{2}{x}\right](x-\beta)$$

$$x = \frac{r}{a_T t}$$

There is a transition point when $x = 1$ (i.e. when $r = a_T t$)

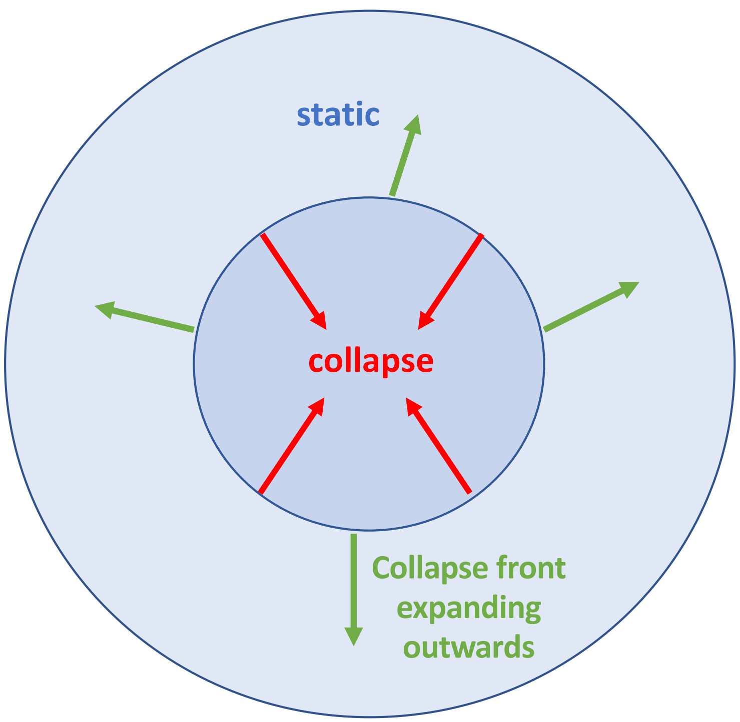
$x < 1$:

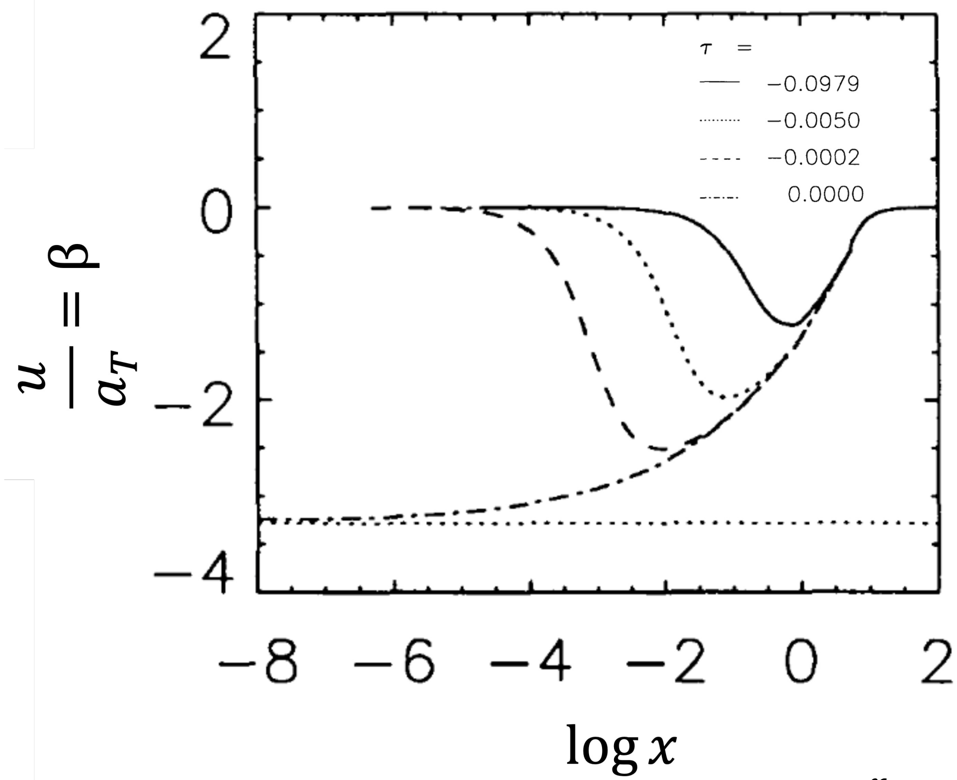
$a_T t > r$
then $\beta < 0$
=> Infall

$x > 1$:

$a_T t < r$
then $\beta > 0$
=> Isothermal sphere

Summary of Monday's lecture





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