Gravitational stability and instability

Chapter 3

Equations of hydrodynamics and hydrostatic equilibrium

TOOLBOX FOR THIS CHAPTER

Euler's equation governs adiabatic and inviscid flow :

$$\rho\frac{d\vec{v}}{dt} + \rho(\overrightarrow{v}\cdot\nabla)\vec{v} = -\nabla P - \rho\nabla\Phi_g$$

Density of the fluid

Flow velocity

Pressure

Gravitational Potential

In equilibrium $v = 0$, then :

$$-\nabla P - \rho\nabla\Phi_g = 0$$

Poisson's equation gives the gravitational potential :

$$\nabla^2\Phi_g = 4\pi G\rho$$

The equation of state for an ideal gas gives the pressure :

$$P = \frac{\rho k_B T}{\mu}$$

The equation of continuity

$$\frac{d\rho}{dt} + \nabla.(\rho v) = 0$$

We also need an equation describing the energy flux but if we use the general form, we will not be able to solve the equations analytically. Approximations are needed !

The isothermal sphere

The simplest model in which pressure and gravity allow stable configuration is the isothermal sphere.

In this model, we assume :
- a spherical symmetry
- a gas at a uniform temperature T
- the equation of state given by : $P = \rho \frac{k_B T}{\mu} = a_T^2 \rho$

where a_T^2 is the isothermal sound speed.

In spherical coordinates (r, θ, φ), the gradient is given by :

$$\nabla f = \frac{\partial f}{dr}\vec{r} + \frac{1}{r}\frac{\partial f}{d\theta}\vec{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{d\varphi}\vec{\varphi}$$

In spherical coordinates (r, θ, φ), the Laplace operator is given by :

$$\Delta f = \nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial f}{d\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$$

Hydrostatic equilibrium
$(v = 0)$

$$\rho\underbrace{\frac{d\vec{v}}{dt}}_{0} + \rho\underbrace{(\overrightarrow{v}\cdot\nabla)\vec{v}}_{0} = -\nabla P - \rho\nabla\Phi_g$$

The gravitational potential and the pressure are only function of the radius r, then the Euler's equation can be written as :

$$\boxed{-\frac{1}{\rho}\frac{dP}{dr} - \frac{d\Phi_g}{dr} = 0}$$

and the Poisson's equation :

$$\boxed{\frac{1}{r^2}\frac{d}{dr}r^2\frac{d\Phi_g}{dr} = 4\pi G\rho}$$

$$\nabla^2\Phi_g = 4\pi G\rho$$

The isothermal sphere

SINGULAR ISOTHERMAL SPHERE

In the following we assume an isolated single isothermal sphere, and we will try to get the solution (pressure, mass, radius)

The gravitational force is given by :

$$-\rho\nabla\Phi_g = -\frac{GM\rho}{r^2}$$

$$-\frac{1}{\rho}\frac{dP}{dr} - \frac{d\Phi_g}{dr} = 0$$

Then, from the Euler's equation we get :

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2}$$

where the mass M, is the mass within a radius r :

$$M = M(r) = \int_0^r \rho(r')4\pi r'^2 dr'$$

The equation of state ($P = \rho\frac{k_B T}{\mu} = a_T^2\rho$) can be written as :

$$\frac{d\rho}{dr} = -\frac{GM}{a_T^2}\frac{\rho}{r^2}$$

Then :

$$\frac{d\ln\rho}{dr}$$

$$r^2\frac{1}{\rho}\frac{d\rho}{dr} = -\frac{GM}{a_T^2}$$

Taking the radius derivative gives :

$$\frac{d}{dr}\left(r^2\frac{d\ln\rho}{dr}\right) = -\frac{G}{a_T^2}\frac{dM}{dr}$$

with $\frac{dM}{dr} = 4\pi r^2\rho$, then

$$\frac{d}{dr}\left(r^2\frac{d\ln\rho}{dr}\right) = -\frac{G}{a_T^2}4\pi r^2\rho$$

The exact solution of this equation is :

$$\boxed{\rho(r) = \frac{a_T^2}{2\pi G r^2}}$$

The isothermal sphere

SINGULAR ISOTHERMAL SPHERE

From the previous equation, we can determine :

- **The total mass of the cloud**

$$\mathrm{M(r_0)} = \int_0^{r_0} \frac{a_T^2}{2\pi G r^2} 4\pi r^2 dr = \frac{2a_T^2 r_0}{G}$$

- In equilibrium, there must be an **external pressure** equaling the pressure at the surface of the cloud :

$$P_0 = a_T^2 \rho(r_0) = \frac{a_T^4}{2\pi G r_0^2}$$

- The **cloud radius** and **isothermal sound speed** can be estimated from the Mass and external pressure
- Although the density and pressure diverge at $r \to 0$, the total mass, internal energy, etc.. are bounded

$$\rho(r) = \frac{a_T^2}{2\pi G r^2}$$

$$M(r) = \int_0^r \rho(r') 4\pi r'^2 dr'$$

Singular sphere because the density and pressure diverge at r=0

The isothermal sphere

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$$\rho(r) = \frac{a_T^2}{2\pi G r^2}$$

$$M(r) = \int_0^r \rho(r') 4\pi r'^2 dr'$$

Singular sphere because the density and pressure diverge at r=0

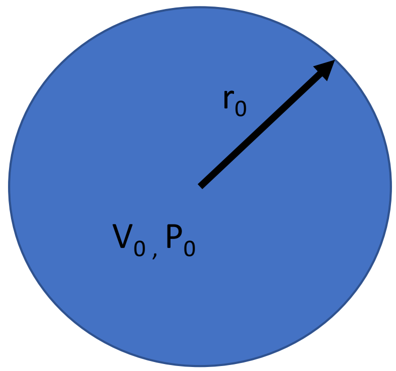
Summary of the isothermal sphere

$$\rho \frac{d\vec{v}}{dt} + \rho(\overrightarrow{v} \cdot \nabla)\vec{v} = -\nabla P - \rho \nabla \Phi_g$$

Hydrostatic equilibrium
Spherical coordinates

$$-\frac{1}{\rho}\frac{dP}{dr} - \frac{d\Phi_g}{dr} = 0$$

Singular Isothermal sphere
T=cst



$$\nabla^2 \Phi_g = 4\pi G \rho$$

Hydrostatic equilibrium
Spherical coordinates

$$\frac{1}{r^2}\frac{d}{dr} r^2 \frac{d\Phi_g}{dr} = 4\pi G \rho$$

$$\rho(r) = \frac{a_T^2}{2\pi G r^2}$$

$$P_0 = a_T^2 \rho(r_0) = \frac{a_T^4}{2\pi G r_0^2}$$

These solutions ($\rho(r)$ and $\mathrm{P}(r)$) diverge at $r \to 0$
Good description of the problem when $r \gg 0$

The isothermal sphere

GENERAL SOLUTION

The Euler's equation gives :

$$-\frac{1}{\rho}\frac{dP}{dr} - \frac{d\Phi_g}{dr} = 0$$

with $P = a_T^2\rho$, then :

$$-\frac{a_T^2}{\rho}\frac{d\rho}{dr} = \frac{d\Phi_g}{dr}$$

Integrating gives :

$$-\ln\rho = \frac{\Phi_g}{a_T^2} + cste$$

Or :

$$\rho(r) = \rho_c \exp\left(-\frac{\Phi_g(r)}{a_T^2}\right)$$

$\rho_c = \rho(r = 0) \neq 0$

$$\nabla^2\Phi_g = 4\pi G\rho$$

To simplify the analysis, we need to introduce dimensionless variables :

$$\psi = \frac{\Phi_g}{a_T^2} \qquad and \;\; \xi = \left(\frac{4\pi G\rho_c}{a_T^2}\right)^{1/2} r$$

The the Poisson's equation becomes :

$$\frac{1}{\xi^2}\frac{d}{d\xi}\xi^2\frac{d\psi}{d\xi} = e^{-\psi}$$

The solution of previous equation is :

$$\psi = \ln\left(\frac{\xi^2}{2}\right)$$

The isothermal sphere

GENERAL SOLUTION

The boundary solutions we can assume are :

- No gravitational force at the center of the cloud :

$$\left(\frac{d\psi}{d\xi}\right)_{\xi=0} = 0$$

- The density at the center of the cloud must be ρ_c, then $\psi(\xi = 0) = 0$

No analytical solution, equation must be integrated numerically.

However, we can estimate the total mass of the cloud :

$$M(r_0) = \int_0^{r_0} \rho 4\pi r^2 dr$$

Introducing the dimensionless variables gives :

$$M(r_0) = 4\pi\rho_c \left(\frac{a_T^2}{4\pi G\rho_c}\right)^{3/2} \int_0^{\xi_0} e^{-\psi}\xi^2 d\xi$$

Then :

$$\boxed{M(r_0) = 4\pi\rho_c \left(\frac{a_T^2}{4\pi G\rho_c}\right)^{\frac{3}{2}} \left(\xi^2 \frac{d\psi}{d\xi}\right)_{\xi=\xi_0}}$$

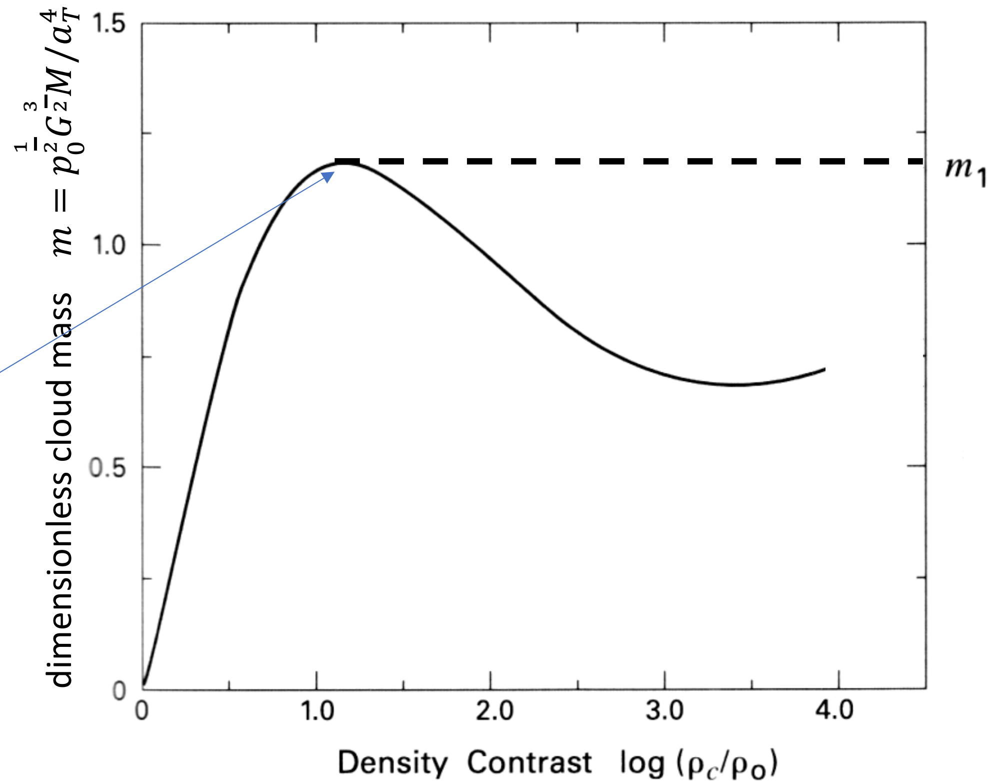
There is one more parameter compared to the singular solution : ρ_c , and we need to also specify a_T^2 , r_0

The isothermal sphere

GENERAL SOLUTION

We can also study the mass of the cloud as a function of the density contrast defined as ρ_c/ρ_0 , where $\rho_0 = \rho(r_0)$

There is a maximum cloud mass (m_1) for which equilibrium can be reached.



The polytropic sphere

The equation state of the polytropic sphere is :

$$P = \mathrm{K}\rho^{1+\frac{1}{n}} = \mathrm{K}\rho^{\Gamma}$$

where n is the polytropic index :

- n=0 for rocky planets
- n=1.5 for star cores

For the general polytropic case, we will demonstrate in problem sheet that the temperature always follows the gravitational potential :

$$k_B T = \frac{1-\Gamma}{\Gamma}\mu\Phi_g$$

Virial equilibrium for the self-gravitating sphere

In the following, we will test if the solutions we find for the isothermal sphere are stable.

The equations of hydrostatic equilibrium are :

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2}$$

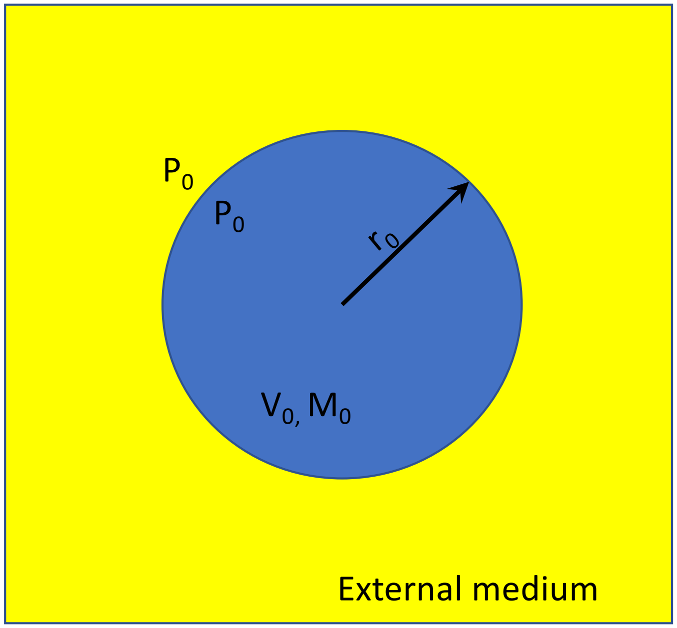
and

$$\frac{dM}{dr} = 4\pi r^2 \rho$$

To simplify, we will consider the mass as an independent variable, such as : $dr = \frac{dM}{4\pi r^2 \rho}$, then:

$$4\pi r^3 dP = -4\pi r\ GM\rho dr = -\frac{GM}{r} dM$$

$$\Rightarrow \quad \int_{V=0,p=p_c}^{V=V_0,p=p_0} 3V\ dP = -\int_0^{M_0} \frac{GM}{r} dM$$



Virial equilibrium for the self-gravitating sphere

$$\int_{V=0,p=p_c}^{V=V_0,p=p_0} 3V\,dP = -\int_0^{M_0} \frac{GM}{r} dM$$

To solve the previous equation, we need to integrate by part :

$$\int_a^b u(x)v'(x)dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)$$

Therefore, we obtain :

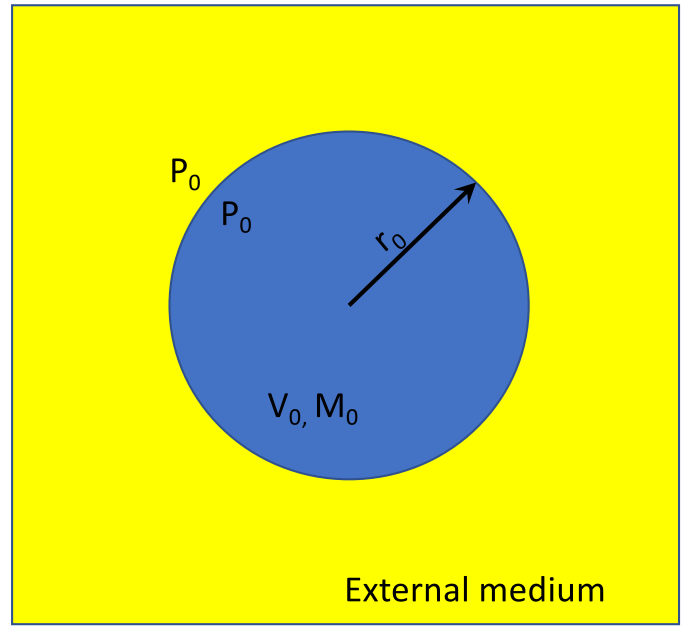
$$3[PV]_{0,p_c}^{V_0,p_0} - 3\int_{0,p_c}^{V_0,p_0} PdV = -\int_0^{M_0} \frac{GM}{r} dM$$

$$3P_0V_0 = 4\pi r_0^3 P_0$$

Gravitational potential (Ω)

$$\boxed{3\int_0^{M_0} \frac{P}{\rho} dM + \Omega = 4\pi r_0^3 P_0}$$

Equation of virial equilibrium



Stability of an isothermal cloud

$$\boxed{3\int_0^{M_0} \frac{P}{\rho} dM + \Omega = 4\pi r_0^3 P_0}$$

For an isothermal sphere, we know that : $P = a_T^2 \rho$, hence :

$$3\int_0^{M_0} \frac{P}{\rho} dM = 3a_T^2 M_0 = 3\,\frac{k_B T}{\mu} M_0$$

We also know that :

$$\Omega = -\frac{3}{5}\frac{GM_0^2}{r_0}$$

Then the virial equilibrium equation for an isothermal cloud becomes :

$$3\,\frac{k_B T}{\mu} M_0 - \frac{3}{5}\frac{GM_0^2}{r_0} - 4\pi r_0^3 P_0 = 0$$

Associated with thermal pressure

Associated with gravity

Associated with external pressure

Stability conditions

- If =0 ➔ equilibrium
- If <0 ➔ external pressure and gravity are "stronger" than thermal pressure : **the cloud is collapsing**
- If >0 ➔ the thermal pressure is larger than external pressure and gravity : **the cloud is expanding**.

Stability of an isothermal cloud

The external pressure P_0 is given by :

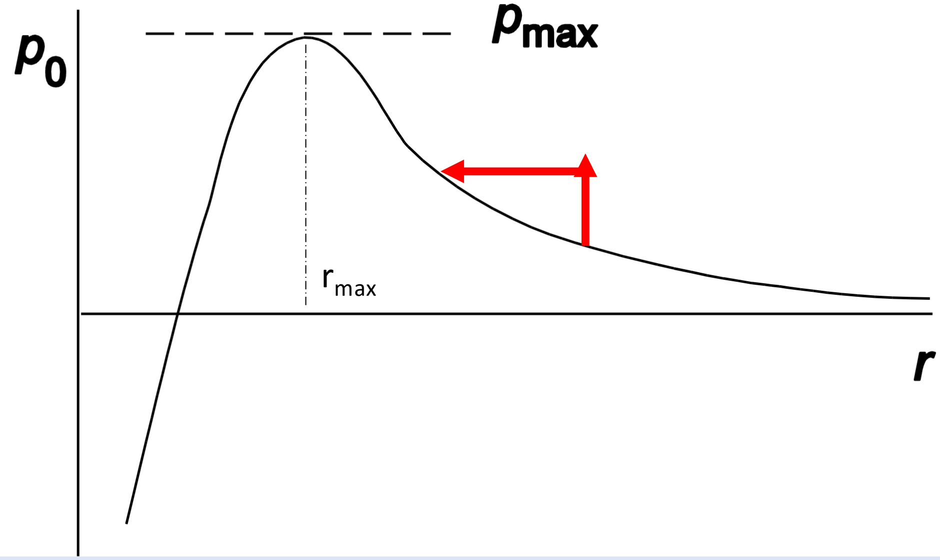
$$P_0(r) = \frac{3k_B T M_0}{4\pi r^3 \mu} - \frac{3}{20\pi}\frac{GM_0^2}{r^4}$$

The maximum of this function is given by :

$$\frac{dP_0(r)}{dr} = r_{max} = \frac{4}{15}\frac{GM_0^2\mu}{k_B T}$$

with a maximum pressure of :

$$P_{max} = c_g \left(\frac{k_B T}{\mu}\right)^4 \frac{1}{G^3 M_0^2}$$



Stability of the cloud of mass M_0 and with $r > r_{max}$:

- If the external pressure is increased by a small amount, the system will lie above the equilibrium line, then the virial equation shows that the cloud must shrink.
- If the external pressure $P_0 > P_{max}$ the cloud is not stable and can't find any radius at which it will be in equilibrium : the cloud is collapsing.

Stability of an isothermal cloud

$$P_{max} = c_g \left(\frac{k_B T}{\mu}\right)^4 \frac{1}{G^3 M_0^2}$$

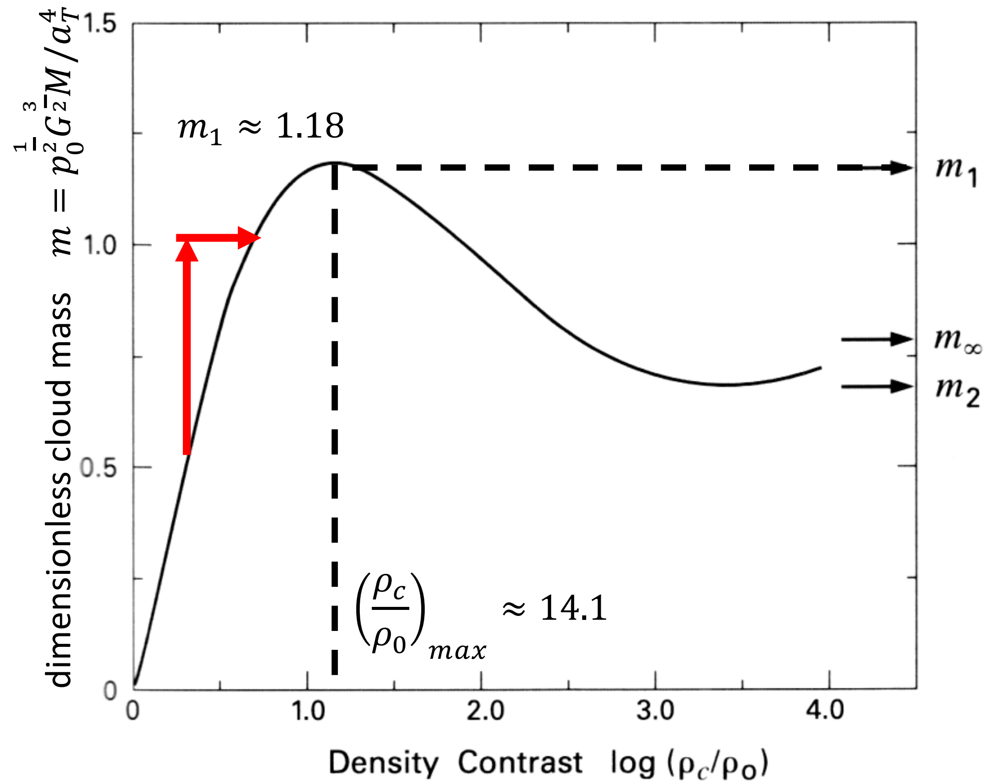
For a given external pressure, a cloud will become unstable to collapse when its mass exceeds :

$$M = c_g^{1/2} \left(\frac{k_B T}{\mu}\right)^2 \frac{1}{G^{3/2} P_0^{1/2}} = c_g^{1/2} \frac{a_T^3}{\rho_0^{1/2} G^{3/2}}$$

Bonnor-Ebert mass

Consider a cloud with a given ρ_c/ρ_0, if we increase the external pressure then :

- The dimensionless mass will increase ($m = p_0^{\frac{1}{2}} G^{\frac{3}{2}} M / a_T^4$)
- For stability the internal pressure of the cloud must increase
- But at constant T, this requires ρ in the cloud to increase and hence ρ_c



Jeans instability

$$\rho \frac{d\vec{v}}{dt} + \rho(\overrightarrow{v} \cdot \nabla)\vec{v} = -\nabla P - \rho \nabla \Phi_g$$

$$\nabla^2 \Phi_g = 4\pi G \rho$$

PERTURBATION ANALYSIS OF FLUID EQUATIONS

Initial conditions of the system (the fluid is stationary) :

- $v_0 = 0$
- $\rho_0 = cste$
- $P_0 = cste$

Introducing perturbed quantities :

- $\rho = \rho_0 + \rho_1$
- $v = v_0 + v_1$
- $\Phi_g = \Phi_0 + \Phi_1$
- $P = P_0 + P_1$

As previously, the unperturbed potential is assumed to satisfy :

$$\nabla^2 \Phi_0 = 4\pi G \rho_0$$

There is no solution when $\rho_0 = cste$

The equation of continuity given by :

$$\frac{d\rho}{dt} + \nabla.(\rho v) = 0$$

becomes (to first order):

$$\rho_0(\nabla.\overrightarrow{v_1}) = -\frac{\partial \rho_1}{\partial t}$$

Similarly, the other hydrostatic equations (Euler's & Poisson's) become :

$$\frac{\partial v_1}{\partial t} = -\nabla \phi_1 - \frac{1}{\rho_0}\nabla P_1$$

And

$$\nabla^2 \phi_1 = 4\pi G \, \rho_1$$

We also assume isothermal behaviour such as :

$$P_1 = a_T^2 \rho_1$$

Jeans instability

$$\rho_0(\nabla.\vec{v_1}) = -\frac{\partial \rho_1}{\partial t}$$

$$\frac{\partial v_1}{\partial t} = -\nabla \phi_1 - \frac{1}{\rho_0}\nabla P_1$$

PERTURBATION ANALYSIS OF FLUID EQUATIONS

Differentiating the continuity equation with respect to time, we obtain :

$$\frac{\partial}{\partial t}(\nabla. v_1) = -\frac{1}{\rho_0}\frac{\partial^2 \rho_1}{\partial t^2}$$

Taking the divergence of the Euler equation :

$$\frac{\partial}{\partial t}(\nabla. v_1) = -\nabla^2 \phi_1 - \frac{a_T^2}{\rho_0}\nabla^2 \rho_1$$

Combining the two previous equation gives :

$$\left(\nabla^2 - \frac{1}{a_T^2}\frac{\partial^2}{\partial t^2} + \frac{4\pi G \rho_0}{a_T^2}\right)\rho_1 = 0$$

→ similar to the wave equation !

Therefore, we should look for wave-like solutions of the form :

$$\rho_1 \propto e^{i(\vec{k}.\vec{r} - \omega t)}$$

which gives a dispersion relation :

$$a_T^2 k^2 - \omega^2 = 4\pi G \rho_0$$

The system is unstable when the modes grow (i.e. $\omega^2 < 0$). Hence we can define a critical wave number (when $\omega^2 = 0$) :

$$k_j^2 = \frac{4\pi G \rho_0}{a_T^2} = \frac{4\pi G \mu}{k_B T}\rho_0$$

And a characteristic wavelength :

$$\lambda_J = \frac{2\pi}{k_j}$$

Jeans instability

PERTURBATION ANALYSIS OF FLUID EQUATIONS

The total mass within a sphere of diameter equal to the Jeans wavelength λ_j is :

$$M_J = \frac{4}{3}\pi\left(\frac{\lambda_j}{2}\right)^3 \rho_0$$

Moreover :

$$\lambda_J^2 = \left(\frac{2\pi}{k_j}\right)^2 = \frac{\pi k_B T}{G\mu\rho_0}$$

Then :

$$\frac{\lambda_J}{2} = \frac{3}{\pi^2}\frac{GM_J\mu}{k_B T}$$

For $\lambda > \lambda_J$ or $M > M_J$ the modes grow exponentially : **the cloud is collapsing**

The Jeans Mass is the mass above which gravity dominates.

The Jeans mass is usually defined as :

$$\frac{M_J}{M_\odot} = 1.0\times\left(\frac{T}{10K}\right)^{3/2} \times \left(\frac{n_H}{2\times10^{10}m^{-3}}\right)^{-1/2}$$

Strong dependance on temperature $\Rightarrow$ *Importance of cooling which could reduce the temperature and therefore allow the collapse of less massive clouds*

Jeans instability

PERTURBATION ANALYSIS OF FLUID EQUATIONS

In the early Universe, the absence of metals and dust (not enough time to form) and the much reduced molecular gas content implies very poor cooling

Formation of massive stars in the early Universe

Magnetic fields

Magnetic fields are important components of the ISM : these can provide additional forces which can act to stabilise clouds against gravitational collapse.

The derivation of the Euler equation in the case of magnetic fields is complex. We will just give the solution of the Euler equation :

$$P_0 = \underbrace{\frac{3k_B T M_0}{4\pi r_0^3 \mu}}_{\text{Thermal Pressure}} + \frac{1}{4\pi r_0^4}\left(\underbrace{\beta \frac{\Phi_M^2}{2\mu_0}}_{\text{Magnetic pressure}} - \underbrace{\frac{3}{5} G M_0^2}_{\text{Gravity}}\right)$$

The pressure will be a monotonically decreasing function of r if :

$$\beta \frac{\Phi_M^2}{2\mu_0} > \frac{3}{5} G M_0^2$$

The clouds will always be stable if Φ_M is a constant.

Application to molecular clouds

Giant Molecular Clouds can be seen as swarms of more coherent clumps.

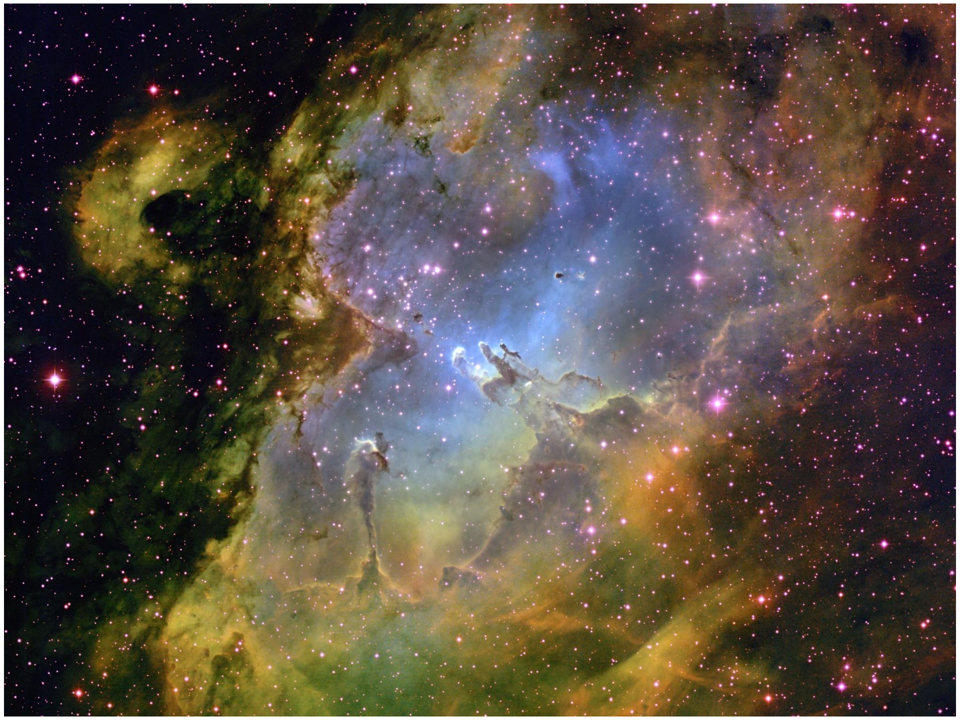
The Jeans mass for gas with $n_H \sim 2000$ and $T \sim 10K$ is $M_J \sim 3M_\odot$. This is well below the observed masses of the individual clouds and of order the mass of typical dense cores.

➔ There must be an additional form of support : the magnetic pressure.

Zeeman splitting provides a method for measuring the magnetic fields in clouds, although this has only be successful in a handful of dark clouds.

From the magnetic virial equation we can find the maximum cloud mass which could be supported against its own self-gravity by magnetic pressure alone :

$$M^2 = \frac{5}{3G}\beta\frac{\pi^2 r^4 B^2}{2\mu_0}$$



Application to molecular clouds

Then :

$$M \approx \left(\frac{B}{nT}\right)\left(\frac{r}{pc}\right)^2 M_\odot$$

Inserting typical values for several cloud types, we can show that most dark clouds can be stabilised by magnetic effects :

- For dense cores $M \sim M_J$
- The density ratio is measured to $\rho_c/\rho_0 \sim 10$

Cloud type	n_{tot} [$10^6 m^{-3}$]	L [pc]	T [K]	M [$M_\odot$]	B [nT]
Giant Molecular Cloud	100	50-500	15	10^5	1 ?
Dark Cloud Complex	500	10	10	10^4	1 ?
Individual Dark Cloud	10^3	2	10	30	2-10
Dense Core	10^4	0.1	10	10	2-10

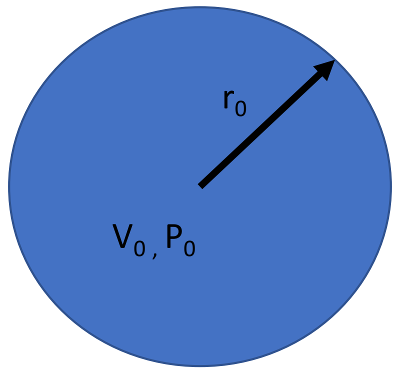
Summary of yesterday's lecture

$$\rho \frac{d\vec{v}}{dt} + \rho(\overrightarrow{v} \cdot \nabla)\vec{v} = -\nabla P - \rho \nabla \Phi_g$$

Hydrostatic equilibrium
Spherical coordinates

$$-\frac{1}{\rho}\frac{dP}{dr} - \frac{d\Phi_g}{dr} = 0$$

Singular Isothermal sphere
T=cst



$$\nabla^2 \Phi_g = 4\pi G \rho$$

Hydrostatic equilibrium
Spherical coordinates

$$\frac{1}{r^2}\frac{d}{dr} r^2 \frac{d\Phi_g}{dr} = 4\pi G \rho$$

$$\rho(r) = \frac{a_T^2}{2\pi G r^2}$$

$$P_0 = a_T^2 \rho(r_0) = \frac{a_T^4}{2\pi G r_0^2}$$

These solutions ($\rho(r)$ and $\mathrm{P}(r)$) diverge at $r \to 0$
Good description of the problem when $r \gg 0$

Summary of yesterday's lecture

$\rho_c = \rho(r=0) \neq 0$

$$\rho(r) = \rho_c \exp\left(-\frac{\Phi_g(r)}{a_T^2}\right)$$

Introducing dimensionless variables

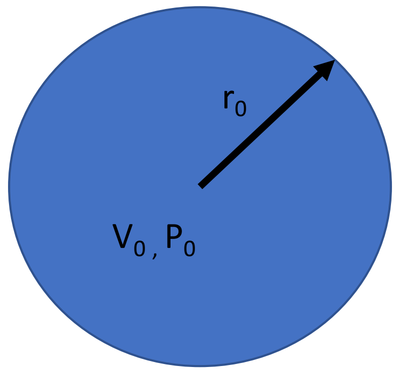
$$\psi = \frac{\Phi_g}{a_T^2} \quad and \ \ \xi = \left(\frac{4\pi G\rho_c}{a_T^2}\right)^{1/2} r$$

Euler's equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\xi^2\frac{d\psi}{d\xi} = e^{-\psi}$$

No analytical solutions ;
numerical integration needed

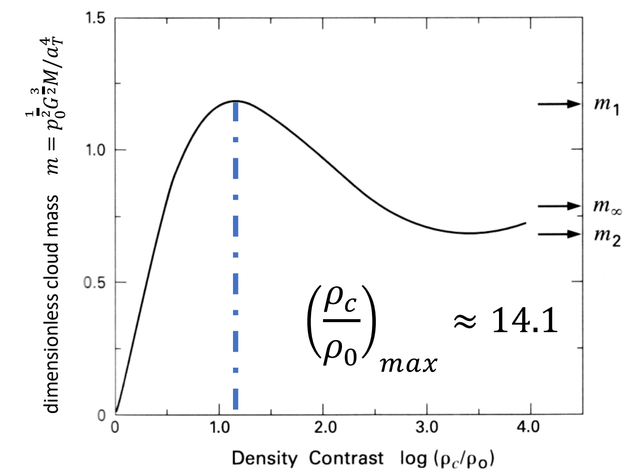
Isothermal sphere
General Solution



$$M(r_0) = 4\pi\rho_c\left(\frac{a_T^2}{4\pi G\rho_c}\right)^{\frac{3}{2}}\left(\xi^2\frac{d\psi}{d\xi}\right)_{\xi=\xi_0}$$

$$\mathrm{M(r_0)} = \frac{2a_T^2 r_0}{G}$$

Comparing with the mass expression obtained from the singular isothermal sphere analysis, we see that one more parameter is needed ρ_c



Summary of yesterday's lecture

Virial equation for a spherical gas cloud

$$3\frac{k_B T}{\mu}M_0 - \frac{3}{5}\frac{GM_0^2}{r_0} - 4\pi r_0^3 P_0 = 0$$

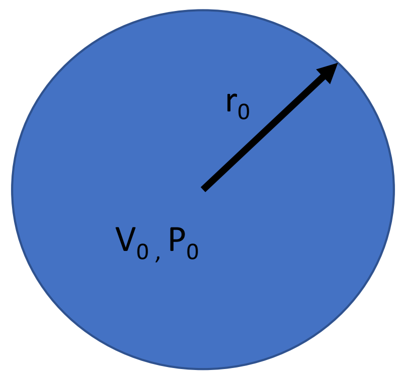
Associated with thermal pressure

Associated with gravity

Associated with external pressure

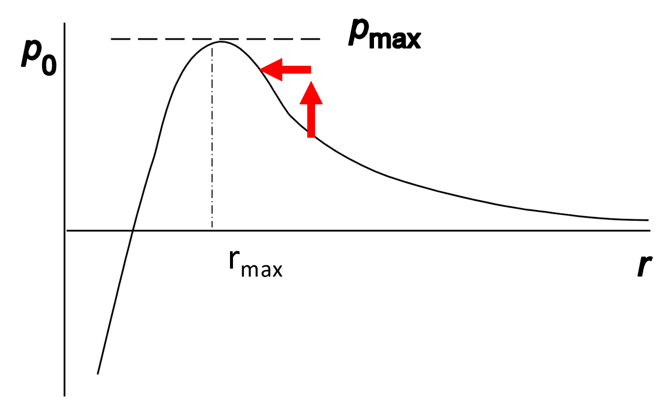
- If =0 ➔ equilibrium
- If <0 ➔ **the cloud is collapsing**
- If >0 ➔ **the cloud is expanding**.

Stability of a gas cloud



Evolution of the pressure

$$P_0(r) = \frac{3k_B T M_0}{4\pi r^3 \mu} - \frac{3}{20\pi}\frac{GM_0^2}{r^4}$$



Summary of yesterday's lecture

Jeans Instability

Introducing perturbed quantities

$$\rho = \rho_0 + \rho_1$$
$$v = v_0 + v_1$$
$$\Phi_g = \Phi_0 + \Phi_1$$
$$P = P_0 + P_1$$

$$\frac{d\rho}{dt} + \nabla.(\rho v) = 0$$

$$\rho \frac{d\vec{v}}{dt} + \rho(\overrightarrow{v} \cdot \nabla)\vec{v} = -\nabla P - \rho \nabla \Phi_g$$

$$\nabla^2 \Phi_g = 4\pi G \rho$$

Only keeping first order terms

$$\rho_0(\nabla.\overrightarrow{v_1}) = -\frac{\partial \rho_1}{\partial t}$$

$$\frac{\partial v_1}{\partial t} = -\nabla \phi_1 - \frac{1}{\rho_0}\nabla P_1$$

$$\nabla^2 \phi_1 = 4\pi G\, \rho_1$$

$$\left(\nabla^2 - \frac{1}{a_T^2}\frac{\partial^2}{\partial t^2} + \frac{4\pi G \rho_0}{a_T^2}\right)\rho_1 = 0$$

Summary of yesterday's lecture

The Jeans Mass is the Mass above which gravity dominates => collapse ! Look at the strong dependence in temperature, and hence the crucial role of cooling (previous chapter) in the formation of structure !

$$\frac{M_J}{M_\odot} = 1.0 \times \left(\frac{T}{10K}\right)^{3/2} \times \left(\frac{n_H}{2\times10^{10} m^{-3}}\right)^{-1/2}$$

Wave equation

$$\left(\nabla^2 - \frac{1}{a_T^2}\frac{\partial^2}{\partial t^2} + \frac{4\pi G\rho_0}{a_T^2}\right)\rho_1 = 0$$

Form of the solution

$$\rho_1 \propto e^{i(\vec{k}.\vec{r}-\omega t)}$$

Jeans Mass

$$M_J = V_0\rho_0 = \frac{4}{3}\pi\left(\frac{\lambda_j}{2}\right)^3\rho_0$$

Dispersion equation

$$a_T^2 k^2 - \omega^2 = 4\pi G\rho_0$$

Critical Wave Number

$$k_j^2 = \frac{4\pi G\rho_0}{a_T^2} = \frac{4\pi G\mu}{k_B T}\rho_0$$

Characteristic Wavelength

$$\lambda_J = \frac{2\pi}{k_j}$$

$\omega = 0$

<u>Question</u> : spectra if we have as many emissions as absorptions